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A COUNTEREXAMPLE TO THE BISHOP–PHELPS THEOREM IN COMPLEX SPACES

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ABSTRACT

The Bishop–Phelps Theorem asserts that the set of functionals which attain the maximum value on a closed bounded convex subset S of a real Banach space X is norm dense in X^*. We show that this statement cannot be extended to general complex Banach spaces by constructing a closed bounded convex set with no support points.

Introduction

If S is a subset of a Banach space X, then a nonzero functional $\varphi \in X^*$ is a support functional for S and a point $x \in S$ is a support point of S if $|\varphi(x)| = \sup_{y \in S} |\varphi(y)|$.

In 1958 [4] Victor Klee asked if each closed bounded convex subset of a Banach space must have a support point.

In 1961 E. Bishop and R. R. Phelps in their fundamental paper [1] proved that the set of support functionals for a closed bounded convex subset S of a real Banach space X is norm dense in X^*, thereby verifying Klee's conjecture. They also showed that the same theorem is true if the set S is the unit ball of a complex Banach space. The natural question of whether this statement is true in a complex Banach space for any closed bounded convex subset was left open since then [6]. In 1977 Jean Bourgain [2] proved the remarkable result that the Bishop–Phelps Theorem is correct if a complex Banach space X has the Radon–Nikodym property.

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In this note we construct a complex Banach space X and a closed bounded convex subset S of X such that the set of the support points of S is empty.

This shows that in general Banach spaces the complex version of Klee's Conjecture is false. In particular, it means that the Bishop–Phelps Theorem cannot be extended to Complex Spaces.

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The counterexample

Let $\mathcal{H}^\infty$ be the algebra of analytical functions bounded on the open unit disk $\mathcal{D}$ with the norm $\|f\| = \sup_{z\in\mathcal{D}} |f(z)|$ and with the identity function E. Each point $z \in \mathcal{D}$ defines a point evaluation functional $\varphi_z(f) = f(z)$. It is well known that $\mathcal{H}^\infty$ may be identified with the dual space of some Banach space X such that each functional φ_z is an element in X [3]. We use the notation $<,>$ to describe a scalar product between a Banach space and its dual. Let S be the closed convex hull of the elements $\{\varphi_z\}$. Then obviously for each point $s \in S$ and each function $f \in \mathcal{H}^\infty$ we have

$$\|s\| \leq 1, \langle s, E\rangle = 1 \tag{1}$$

and

$$\sup_{s\in S} |\langle s, f\rangle| = \|f\|. \tag{2}$$

LEMMA 1: *Suppose that $f \in \mathcal{H}^\infty$ and $\|f\| \leq 1$. Then either $f = \lambda E$, $|\lambda| = 1$ or*

$$\lim_{k\to\infty} \langle s, f^k\rangle = 0 \tag{3}$$

for any point $s \in S$.

Proof: Suppose that $f \neq \lambda E$. Since finite convex linear combinations of point evaluations are dense in the set S and the sequence $\{f^k\}$ is bounded by norm we need only consider the case

$$s = \sum_{i=1}^{n} \alpha_i \varphi_{z_i}$$

where $\alpha_i \geq 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$. Put $\theta = \sup_{1\leq i\leq n} |f(z_i)|$. Then

$$|\langle s, f^k\rangle| \leq \sum_{i=1}^{n} \alpha_i |f^k(z_i)| \leq \theta^k \sum_{i=1}^{n} |\alpha_i| = \theta^k.$$

Since $\|f\| \leq 1$ the maximum modulus principle implies that $\theta < 1$, which implies (3). ∎

LEMMA 2: *Suppose there exists an element $s \in S$ and a function $f \in \mathcal{H}^\infty$ such that $\langle s, f\rangle = \|f\| = 1$. Then*

$$\langle s, f^n\rangle = 1 \tag{4}$$

for any positive integer n.

Proof: Let M be the space of maximal ideals of the algebra $\mathcal{H}^\infty$ and let $C(M)$ be the algebra of all continuous functions on M with the sup-norm. The algebra $\mathcal{H}^\infty$ is a Banach subalgebra of the algebra $C(M)$. Let $\hat{s}$ be a norm preserving extension of the functional s onto the space $C(M)$. By the Riesz theorem there exists a regular Borel measure $d\nu$ on M such that the equality

$$\langle s, g\rangle = \int_M g\, d\nu$$

holds for any function $g \in \mathcal{H}^\infty$. The conditions (1) imply that $\int_M d|\nu| \leq 1$ and $\int_M d\nu = 1$. It means that the measure $d\nu$ is a nonnegative probability measure on M. This implies that the function f is equal to the identity function on the support of the measure $d\nu$, which implies (4). ∎

THEOREM 1: *Suppose that the modulus of the functional $g \in \mathcal{H}^\infty$ attains its maximum on the set S. Then there exists a complex number α such that $g = \alpha E$.*

Proof: There exists an element $s_0 \in S$ such that

$$| < s_0, g > | = \sup_{s \in S} | < s, g > |.$$

From (2) we have that $| < s_0, g > | = ||g||$. If g is the zero function, then we can put $\alpha = 0$. If g is a nonzero function, then there exists a complex number λ such that $< s_0, \lambda g >= ||\lambda g|| = 1$. Put $f = \lambda g$. Then (4) and Lemma 1 implies that $f = \gamma E$ and $g = \lambda f = \lambda\gamma E$. ∎

So the line αE in $\mathcal{H}^\infty$ is the set of all support functionals for the closed bounded convex subset S in the predual space X of $\mathcal{H}^\infty$.

Let φ_0 be the point evaluation at 0 and let L be the line in X generated by φ_0. Let X_1 be the quotient space X/L and π_1: $X \to X_1$ be the corresponding quotient map. The dual space X_1^* is the annihilator $\varphi_0^\perp$ of the vector φ_0 in the space X^* which is the hyper-plane H_0^∞ of all functions in H^∞ vanishing at 0. Put $S_1 = \pi_1(S)$. Then obviously the set S_1 is closed bounded and convex.

THEOREM 2: *The set of support points of S_1 is empty.*

Proof: Since the equality $< \pi_1(s), f >=< s, f >$ holds for any point $s \in S$ and any functional $f \in \varphi_0^\perp$, Theorem 1 implies that the only possible support functional for the set S_1 is a functional λE. Since $< \varphi_0, E >= 1$, the line λE has zero intersection with the subspace $\varphi_0^\perp$.

For this reason the only one functional which attains a maximum modulus on the set S_1 is the zero functional and the set of support points of the set S_1 in the predual to the space H_0^∞ is empty. ∎

References

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